A note on the special case of full rank:
Consider $\mathbf{A}$, an n×n matrix, where the Rank($\mathbf{A}$) = n. Matrix $\mathbf{A}$ has the following characteristics:

- Full rank
- Nonsingular
- Linearly independent
- Determinant $\neq 0$
- Can invert $\mathbf{A}$
- Can solve equations
- Unique solution

Normalizing

Normalizing is a process that is helpful when conducting multivariate statistical procedures. Often, we have variables that are measured on very different scales. Consider GPA, GRE scores, and self-efficacy scores. These will typically be measured on very different scales. To place them on a common scale for some comparative purposes, we say that we can standardized them, as in computing z-scores. In matrix algebra, this process is called normalizing.

Recall that the inner-product (or scalar product) of two vectors, such as $\underline{x}'\underline{x}$ can provide special information. In one case, it is related to the length of the vector, since $L(\underline{x}) = \sqrt{\underline{x}'\underline{x}}$. In another case, $\underline{d}'\underline{d}$ can provide the sum-of-squares, where $\underline{d}$ is a deviation vector.

If $\underline{a}'\underline{b} = a_1b_1 + a_2b_2 + \ldots + a_nb_n = 0$, then $\underline{a}$ and $\underline{b}$ are orthogonal. The $\text{Cos}(\theta_{ab})=0$.

If $\underline{a}'\underline{a} = 1$, $\underline{a}$ is normalized. **Normalized vectors have a length of 1.**

This is also called a unit vector.

We can normalize a vector: $\underline{x}^* = \dfrac{1}{\sqrt{\underline{x}'\underline{x}}}\underline{x}$. Essentially, multiplying a vector by the inverse of its length normalizes the vector.

We can prove how the inner-product of a normalized vector equals 1. Replacing each component of the inner product $\underline{a}'\underline{a}$ with the normalized vector ($\underline{x}^*$), and factoring out the common inverse of the length, we see that the result is 1:

$$\underline{a}'\underline{a} = 1 \rightarrow \left(\frac{1}{\sqrt{\underline{x}'\underline{x}}}\underline{x}\right)'\left(\frac{1}{\sqrt{\underline{x}'\underline{x}}}\underline{x}\right) = \left(\frac{1}{\sqrt{\underline{x}'\underline{x}}}\right)^2 \underline{x}'\underline{x} = \frac{1}{\underline{x}'\underline{x}}\ \underline{x}'\underline{x} = 1$$

Canonical Correlation

Recall the formula for the correlation, which is the standardized covariance:

$$r = \frac{\text{cov}(x, y)}{\sqrt{\text{var}(x)}\sqrt{\text{var}(y)}} \text{ and } r^2 = \frac{\text{cov}(x, y)^2}{\text{var}(x)\,\text{var}(y)} = \text{cov}(x,y)^2\ \text{var}(x)^{-1}\ \text{var}(y)^{-1}.$$

Introduced by Hotelling in 1936, the canonical correlation is a measure of the association between two sets of variables, whereas a bivariate correlation is the association between two single variables. So the canonical correlation is the multivariate analogue to the bivariate correlation.

Although it is much more complicated than we will explore here, canonical correlation analysis determines a set of canonical values, orthogonal linear combinations of the variables within each set that best explains the variability within and between the two sets of variables. This is in the case where there are multiple Y variables (math scores, reading scores, science scores) and X variables (GPA, gender, SES). We might ask for a multivariate correlation between the two sets of variables. So we might ask about the magnitude of association between the background variables (Xs) and the achievement variables (Ys). We can square this value and obtain a very useful statistic that is essential to multivariate factor models, the eigen value.

Canonical correlations do not require that one set of variables be considered dependent or independent. It is not a hypothesis testing method, but is a descriptive and exploratory technique.

By rearranging the terms in the result above for the bivariate correlation, to ensure that the matrices are conformable, we obtain:

$$\text{var}(y)^{-1}\ \text{cov}(x,y)\ \text{var}(x)^{-1}\ \text{cov}(x,y) = \mathbf{\Sigma}_{yy}^{-1}\mathbf{\Sigma}_{xy}\mathbf{\Sigma}_{xx}^{-1}\mathbf{\Sigma}_{xy}$$

This is sometimes written slightly differently in arrangement of variance-covariance matrices and sometimes with correlation matrices. The canonical correlation is the correlation between a single linear function of the Ys and a single linear function of the Xs. This requires estimating the characteristic roots (eigenvalues) of the solution above, which are the squared canonical correlation coefficients. We will see these terms in the next notes.

Canonical correlation squared = λ, the eigenvalue.

Here is a link to a website on how to get started with canonical correlation analysis:

http://jeromyanglim.blogspot.com/2010/06/canonical-correlation-getting-started.html

A note regarding Row & Column Rank Equivalence

Consider an n × p matrix.

The n rows of the matrix are vectors in R^n, spanning the subspace called row space. The dimension of this space is called row rank.

The p columns of the matrix are vectors in R^p, spanning the subspace called column space. The dimension of this space is called column rank.

A theorem of matrix algebra proves row rank = column rank = rank.

Rank of an n × p matrix = rank of largest submatrix where $|\mathbf{A}| \neq 0$.

$\mathbf{A}$ is full rank if rank($\mathbf{A}$) = min{n,p}.